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Holographic RG flows and universal structures on the Coulomb branch of $\mathcal{N}=2$ supersymmetric large-N gauge theory

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Abstract: We report on our results of D3-brane probing a large class of generalised type IIB supergravity solutions presented very recently in the literature. The structure of the solutions is controlled by a single non-linear differential equation. These solutions correspond to renormalisation group flows from pure $\mathcal{N}$=4 supersymmetric gauge theory to an $\mathcal{N}$=2 gauge theory with a massive adjoint scalar. The gauge group is SU(N) with N large. After presenting the general result, we focus on one of the new solutions, solving for the specific coordinates needed to display the explicit metric on the moduli space. We obtain an appropriately holomorphic result for the coupling. We look for the singular locus, and interestingly, the final result again manifests itself in terms of a square root branch cut on the complex plane, as previously found for a set of solutions for which the details are very different. This, together with the existence of the single simple non-linear differential equation, is further evidence in support of an earlier suggestion that there is a very simple model — perhaps a matrix model with relation to the Calogero-Moser integrable system — underlying this gauge theory physics.

Keywords: Duality in Gauge Field Theories, Superstring Vacua, Brane Dynamics in Gauge Theories, AdS-CFT and dS-CFT Correspondence.

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1. Introductory remarks

Last week, a new solution to typeIIB supergravity was presented in ref. [1]. It beautifully clarified and considerably generalised the structure of an already interesting family of supergravity solutions presented in ref. [2] (see also ref. [3].) Those earlier supergravity solutions are asymptotic to the maximally supersymmetric $\mathrm{AdS}_5 \times S^5$ solution, and *via* the AdS/CFT correspondence [4, 5], and generalisations thereof, represent "Holographic Renormalisation Group Flow" [6, 7] of the four dimensional $\mathcal{N}$=4 supersymmetric pure Yang-Mills theory to $\mathcal{N}$=2 supersymmetric gauge theory with a massive hypermultiplet in the adjoint representation of $\mathrm{SU}(N)$, in the large-N limit. These new, more general solutions preserve the same supersymmetries, and are also believed to represent the same type of physics.

In this short note we carry out a study in the spirit of refs. [8, 9]. There were puzzling singularities in the supergravity solutions of ref. [2] obscuring the gauge theory interpretation considerably [2, 3]. The idea was to probe the geometry with the most natural object to hand — one of the constituent D3-branes — in an effort to determine the correct physics. This had borne considerable fruit in a study reported in ref. [10], where the nature of the singular behaviour was understood, unphysical singularities were removed, and the new phenomenon called the "enhançon mechanism" proved to be well adapted to the task of clarifying the physics. As the situation in ref. [2] preserved the same supersymmetries as the geometries under discussion in ref. [10] (where in fact the motivation was also to find gauge duals of $\mathcal{N}$=2 four dimensional gauge theory), it was very natural to bring the same tools to understanding the geometries of ref. [2]. Those studies were successful, and in light of the existence of the more general class of solutions presented recently in ref. [1], it is natural to carry out the same study here. We probe the geometries and derive the effective lagrangian for the general form of solutions in section 3.

The detailed form of a solution is seeded by a non-linear differential equation which yields a single function [1]. This is a difficult equation, and so far only two families of solution are known. The first is the previously known family [2, 3]. In section 4, as a review and for contrast, we specialise our result to that case, and recover the results of ref. [8, 9]. We particularly follow the lines of ref. [8] in moving away from the natural supergravity coordinates and finding new coordinates on the moduli space that are adapted to the full discussion of the low energy effective lagrangian of the $\mathcal{N}$=2 gauge theory. In particular, in the spirit of ref. [11], ref. [8] exhibited explicit holomorphy and identified in the new coordinates the locations of the points where there are singularities in the gauge coupling.

In section 5 we carry out a similar analysis for the second family of exact solutions to the differential equation exhibited in ref. [1]. Whilst the solution is very different from the earlier one, we find that once we have identified the natural coordinates, the locus of singular points is controlled by a very similar functional dependence as in the previous example: there is a dense locus of singular points on a straight line segment controlled by an (inverse) square root branch cut in the complex coupling τ.

We find this simplicity to be intriguing, and suggestive of a universality that it would be interesting to prove. The universality itself is in line with a conjecture made some time ago about the existence of a much simpler model which might underlie the physics [13]: Some of the gross features are similar to a reduced dynamical model such as a large-N matrix model or integrable system related to the Calogero-Moser model at large-N. We discuss some of these ideas and features in section 6.

2. The ten dimensional geometry

We first present the complete solution of ref. [1]. The Einstein frame metric is:

$$ds^2 = \Omega^2(k^2\eta_{\mu\nu}dx^\mu dx^\nu)+\Omega^{-2}\{H_1\left[du^2+u^2(\sigma_2^2+\sigma_3^2)\right]+H_1^{-1}u^2\sigma_1^2+H_2dv^2+H_2^{-1}v^2d\phi^2\}\,,$$

where k is a constant, and

$$\begin{aligned}
\eta_{\mu\nu}dx^\mu dx^\nu &= -(dx^0)^2+(dx^1)^2+(dx^2)^2+(dx^3)^2\,,\\
H_1(u,v) &= \frac{1}{\cos\beta}\,,\qquad H_2(u,v)=\frac{1}{c\cos\beta}\,,\\
\Omega(u,v) &= \frac{u^{1/2}}{(H_1-H_1^{-1})^{1/4}}\,,
\end{aligned}\tag{2.1}$$

where $d\sigma_1=2\sigma_2\wedge d\sigma_3$ (and cyclic permutations) define the left invariant Maurer-Cartan forms on S^3, and note that:

$$H_1H_2^{-1}=c\,,\qquad H_1H_2=\partial_v(vc)\,.\tag{2.2}$$

The other supergravity fields of relevance here are the axion-dilaton fields and the R-R four-form potential. These are given as:

$$\tau\equiv C_{(0)}+ie^{-\Phi}=\frac{\tau_0-\bar\tau_0B}{1-B}\,;\qquad B=\left(\frac{1-H_2}{1+H_2}\right)e^{2i\phi}\,,$$

$$C_{(4)} = w(u,v) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \,, \qquad w(u,v) = \frac{k^4}{4g_s} \frac{u^2}{(H_1^2 - 1)} = \frac{k^4}{4g_s} \Omega^4 \cos\beta \,, \tag{2.3}$$

where by setting

$$\tau_0 = \frac{i}{g_s} + \frac{\theta_s}{2\pi} \,, \tag{2.4}$$

we have set the asymptotic value of the dilaton and the R-R scalar $C_{(0)}$ in terms of the string coupling g_s and the constant θ_s. These in turn set the asymptotic Yang-Mills coupling ($g^2_{\rm YM} = 2\pi g_s$) and the theta angle in the SU(N) gauge theory on the D3-branes to which this geometry is holographically dual at large-N.

We also note here that the solution has been presented with the natural length scale, L, of the spacetime, set to unity. It is given in terms of the string tension, string coupling, and N (the number of branes sourcing the geometry) as $L^4 = 4\pi(\alpha')^2 g_s N$. This can easily be restored as needed.

For completeness, we also note that the three-form field strength $G_{(3)}$ is given by:

$$G_{(3)} = (1 - BB^*)^{-1} \left(dA_{(2)} - B dA^*_{(2)} \right) \,, \tag{2.5}$$

where

$$A_{(2)} = e^{i\phi} \left(a_1 dv \wedge \sigma^1 + a_2 \sigma^2 \wedge \sigma^3 + a_3 \sigma^1 \wedge d\phi \right) \,, \tag{2.6}$$

with

$$a_1(u,v) = \frac{i}{c} \,, \qquad a_2(u,v) = i \frac{v}{v \partial_v c + c} \,, \quad a_3(u,v) = -uv \partial_u c + 2vc \,. \tag{2.7}$$

As we shall see, we will not need these fields in the study presented here.

The remarkable thing about this large class of solutions is that it is seeded entirely by the function $c(u,v)$, which is obtained as a solution to the following non-linear differential equation:

$$\frac{\partial}{\partial u} \left(\frac{v^3}{u} \frac{\partial c}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{v^3}{u} c \frac{\partial c}{\partial v} \right) = 0 \,. \tag{2.8}$$

This equation is very difficult to solve exactly, and only two classes of exact solutions (those which we study here) are known at present. It is interesting to note, however, that a perturbative study can reveal some structures that might be of use for either searching for exact solutions or for seeding numerical studies. The point is that (as suggested in ref. [1]) one can write

$$c(u,v) = 1 + \sum_i c_i(u,v) \lambda^i$$

and attempt to determine the functions $c_i(u,v)$ from the resulting linearised equations. This is an interesting line of attack that we have not carried out in great detail so far. At first order we have noticed that if we separate variables according to $c_1(u,v) = U_1(u) V_1(v)$, then $\tilde{U}_1 = U_1/u$ and $\tilde{V}_1 = V_1 v$ both satisfy relations of the form of Bessel's equation. This may be a clue for further study.

3. The general probe result

In the time-honoured fashion, we will probe the supergravity solution of the previous section with a D3-brane, whose world-volume action is:

$$S = -\tau_3 \int_{\mathcal{M}} d^4\xi \;\; \det{}^{1/2}\left[G_{ab} + e^{-\Phi/2}\mathcal{F}_{ab}\right] + \mu_3 \int_{\mathcal{M}} C_{(4)}\,, \tag{3.1}$$

where

$$\mathcal{F}_{ab} = B_{ab} + 2\pi\alpha' F_{ab}\,, \qquad \mu_3 = \tau_3 g_s = (2\pi)^{-3}(\alpha')^{-2}\,,$$

and the spacetime fields are pulled back *via* the map $x^\mu(\xi^a)$ according to e.g.:

$$G_{ab} \equiv \frac{\partial x^\mu}{\partial \xi^a}\frac{\partial x^\nu}{\partial \xi^b} G_{\mu\nu}\;.$$

The D3-brane will be chosen as lying in the directions $\{x^0, x^1, x^2, x^3\}$, and so the "static gauge" will be chosen so as to respectively align the world-volume coordinates $\{\xi^0, \xi^1, \xi^2, \xi^3\}$ with the spacetime coordinates, and the remaining transverse coordinates, denoted generically x^i, will be taken to be functions, $x^i(t)$, of $x^0 = \xi^0 \equiv t$, allowing for the brane's motion.

We keep only terms quadratic in all velocities, looking to stay in the BPS limit in the directions where this is possible. This computation is quite standard [12], and so we just state the result here with no further elaboration. We obtain an effective lagrangian for a point particle moving in the six transverse coordinates $\mathcal{L} = T - V$, where:

$$\begin{aligned} T &= \frac{\mu_3 k^2}{2g_s}\left\{H_1\left[\dot u^2 + u^2(\dot\sigma_2^2 + \dot\sigma_3^2)\right] + H_1^{-1}u^2\dot\sigma_1^2 + H_2\dot v^2 + H_2^{-1}v^2\dot\phi^2\right\}\,,\\ V &= \frac{\mu_3 k^4}{g_s}\Omega^4\left(1-\cos\beta\right) = \frac{\mu_3 k^4}{g_s}\left(\frac{u^2}{H_1+1}\right)\,. \end{aligned} \tag{3.2}$$

4. Specialising to the previous results

It is instructive to first obtain a few known results. The solution of ref. [2] can be recovered by choosing new coordinates (r,θ), and a function ρ such that

$$u(r,\theta) = \frac{\rho^3\cos\theta}{(c^2-1)^{1/2}}\,, \qquad v(r,\theta) = \frac{\sin\theta}{(c^2-1)^{1/2}}\,, \tag{4.1}$$

where the function ρ is related to c in the following way:

$$\rho^6 = c + (c^2-1)\left[\gamma + \frac{1}{2}\log\left(\frac{c-1}{c+1}\right)\right]\,, \tag{4.2}$$

for a real number γ, which parameterises a whole family of solutions. It is very useful to examine the behaviour of the families by looking at figure 1. The functions (α,χ) plotted there are related to the functions (ρ, c) by $(\rho = e^\alpha, c = \cosh(2\chi))$. The function χ determines the mass of the adjoint hypermultiplet in the $\mathcal{N}$=2 gauge theory that results from the soft breaking of the pure $\mathcal{N}$=4 theory. In fact, the mass is related to the constant k in the solution by [8] $m = k$. The function α sets the vacuum expectation of the remaining $\mathcal{N}$=2 complex scalar in the gauge multiplet.

The left hand side represents the ultraviolet (UV): the $\mathcal{N}=4$ theory with these scalars — and hence the coefficients of the operators to which they correspond — switched off. As we flow to the right the scalars switch on. This corresponds to the mass and the vev in the gauge theory increasing as we flow to the infrared (IR). Notice that for $\gamma < 0$, the cases running to $\rho = 0$ all define a particular finite value of c, which we shall denote c_0. For $\gamma = 0$ this value of c_0 diverges.

Equations 4.1 give

$$\rho^6 = \frac{u^2(c^2-1)}{1-v^2(c^2-1)}\,, \tag{4.3}$$

which can be used to eliminate ρ. The resulting equation, after differentiating with respect to u and to v, removing γ, gives a pair of equations for $\partial_u c$ and $\partial_v c$ that imply the non-linear differential equation 2.8.

The function $\cos\beta$ can be written as:

$$\cos\beta = \frac{X_1^{1/2}}{c^{1/2}X_2^{1/2}}\,, \tag{4.4}$$

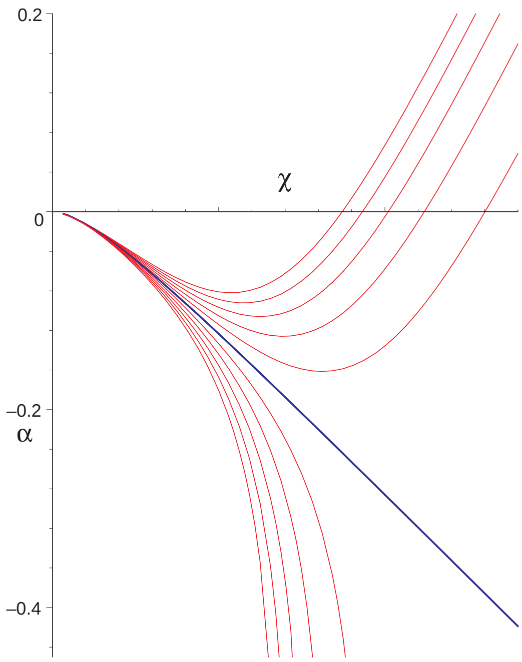

Figure 1: A family of curves of (α, χ). Note that in the text, the natural functions discussed are $(\rho = e^{\alpha}, c = \cosh(2\chi))$. There are three natural classes, $\gamma < 0$ (lying below the central curve), $\gamma = 0$ (the central curve), and $\gamma > 0$ (above the central curve).

where

$$X_1 = \cos^2\theta + \rho^6 c \sin^2\theta\,, \qquad X_2 = c\cos^2\theta + \rho^6\sin^2\theta\,, \tag{4.5}$$

and

$$\Omega^4 = \frac{c^{1/2}\rho^6 X_1^{1/2} X_2^{1/2}}{(c^2-1)^2}\,, \tag{4.6}$$

and so the potential becomes

$$V = \frac{\mu_3 k^4}{g_s}\frac{\rho^6}{(c^2-1)^2}\left[(cX_1X_2)^{1/2} - X_1\right], \tag{4.7}$$

and so there are two separate branches to the moduli space, where the potential vanishes: either $\rho = 0$ or $cX_2 = X_1$. The latter condition is simply $\theta = \pi/2$.

In both branches the moduli space is two dimensional, as appropriate to the fact that we are looking at the moduli space of a single complex scalar component that breaks the SU(N) to SU($N-1$) × U(1), achieved by pulling a single brane away from the collection of the other branes. For the first branch (which only exists for $\gamma < 0$) the coordinates are (θ, ϕ) and the metric on that space is [8]

$$ds_1^2 = \frac{\mu_3 k^2}{2g_s}\frac{1}{c_0^2-1}\left(\cos^2\theta d\theta^2 + \sin\theta^2 d\phi^2\right). \tag{4.8}$$

After changing variables *via* $\sin\theta = r$, this can also be written as:

$$ds_1^2 = \frac{\mu_3 k^2}{2g_s}\frac{1}{c_0^2-1}\left(dr^2 + r^2 d\phi^2\right). \tag{4.9}$$

The parameter r's maximum value is unity, marking the edge of a disc, which is where $\theta = \pi/2$. This edge precisely matches onto the second branch for which $\theta = \pi/2$ everywhere. The natural coordinates on this other branch are (c, ϕ) and the metric is [8, 9]:

$$ds_2^2 = \frac{\mu_3 k^2}{2g_s}\frac{c}{c^2-1}\left(\frac{dc^2}{(c^2-1)^2} + d\phi^2\right). \tag{4.10}$$

Our interest is the vanishing of the metric, corresponding to where singularities on the Coulomb branch appear. This occurs when c diverges, which happens for $\gamma \geq 0$.

The key piece of physics to identify is the location of this singular behaviour in terms of variables that are correctly adapted to the $\mathcal{N}$=2 gauge theory discussion [8]. To find these it is natural [8, 9] to first find coordinates $z = ve^{-i\phi}$ that make the metric conformal to flat space $dzd\bar{z}$:

$$\frac{dc^2}{(c^2-1)^2} = \frac{dv^2}{v^2} \quad\Longrightarrow\quad v = \sqrt{\frac{c+1}{c-1}}. \tag{4.11}$$

The metric is now:

$$ds^2 = \frac{\mu_3 k^2}{2g_s}\frac{c}{(c+1)^2} dz d\bar{z}. \tag{4.12}$$

Next, we must find a coordinate change to a new coordinate Y such that the metric is:

$$ds^2 = \frac{\mu_3 k^2}{2} e^{-\Phi} dY d\bar{Y}. \tag{4.13}$$

The coordinate Y is to represent the vacuum expectation value of the scalar in the gauge multiplet whose moduli space we are examining. The prefactor is the same as the one which appears in front of the kinetic term for the gauge field, and so determines the appropriate coordinates to use. The dilaton can be determined from equations 2.3 to be:

$$e^{-\Phi} = \frac{c}{g_s|\cos\phi + ic\sin\phi|^2}. \tag{4.14}$$

Writing

$$dz d\bar{z} = dY d\bar{Y}\frac{\partial z}{\partial Y}\frac{\partial \bar{z}}{\partial \bar{Y}}, \tag{4.15}$$

we have

$$\left|\frac{\partial Y}{\partial z}\right|^2 = k^2\left|\frac{\cos\phi + ic\sin\phi}{c+1}\right|^2 = \frac{k^2}{4}\left|1 - \frac{1}{z^2}\right|^2, \tag{4.16}$$

where in the last line we have used repeatedly that $z = ve^{-i\phi}$ and that $v = \sqrt{(c{+}1)/(c{-}1)}$. So finally we have the elegant result:

$$Y = \frac{k}{2}\left(z + \frac{1}{z}\right). \tag{4.17}$$

It is in terms of the quantity Y we should look for the non-trivial behaviour of the coupling. To that end, we compute $\tau(Y)$. Using that $\cos\beta = 1$ (from equation 4.4), we have from equations 2.1 and 2.3 that $B = z^{-2}$, and hence (setting $k = m$):

$$\begin{aligned}\tau &= \left(\frac{\tau_0 - \bar{\tau}_0 z^2}{1 - z^2}\right) \\ &= \frac{i}{g_s}\left(\frac{Y^2}{Y^2 - m^2}\right)^{1/2} + \frac{\theta_s}{2\pi}\,.\end{aligned} \tag{4.18}$$

So finally we conclude that there is a singular locus of points where the gauge coupling diverges on the complex Y plane. It is given by the location of the square root branch cut in the function 4.18, and is the large-N analogue of the Seiberg-Witten singular locus for this particular branch of the $\mathcal{N}$=2 supersymmetric SU(N) gauge theory with massive adjoint hypermultiplet of mass m. We will discuss this further in section 6.

5. Another example

In ref. [1], where the new, more general solution (displayed in section 2) was presented, a very simple solution to the differential equation 2.8 that seeds the solution was discovered. This represents a new slice of the Coulomb branch, and we should use the techniques we have been studying to examine it. We can simply specialise our results for the probe computation, and attempt to carry out some of the analysis of the previous section.

The solution is described as "separable", since each half of the non-linear differential equation 2.8 is identically zero:

$$c = \mu(1 + bu^2)\left(1 - \frac{a}{v^2}\right)^{1/2}, \tag{5.1}$$

for a, b, μ arbitrary real constants, which gives:

$$H_1 = \mu(1 + bu^2)\,, \qquad H_2 = \frac{v}{(v^2 - a)^{1/2}}\,. \tag{5.2}$$

After substitution, the potential is:

$$V = \frac{\mu_3 k^4}{g_s}\left(\frac{u^2}{\mu(1 + bu^2) + 1}\right). \tag{5.3}$$

Now, matching to the asymptotic value of c requires that $\mu = 1$ and $b = 0$. Inserting these (the first is enough in fact) shows that there is only one branch for the moduli space, which is $u = 0$. The moduli space is again two dimensional, as expected, and the metric is:

$$ds^2 = \frac{\mu_3 k^2}{2g_s}(v^2 \pm a)^{1/2} v \left(\frac{dv^2}{v^2 \pm a} + d\phi^2\right), \tag{5.4}$$

where we have taken a to mean its positive part and hence written the two choices of sign we can have in the metric.

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We must now find the natural $\mathcal{N}$=2 coordinates. First we write, taking the plus sign (we will deal with the minus case later):

$$ds^2 = \frac{\mu_3 k^2}{2g_s}(v^2+a)^{1/2}v\left(\frac{dw^2}{w^2}+d\phi^2\right), \tag{5.5}$$

for some new radial coordinate we must find, w. After some algebra, we find:

$$v = \frac{a^{1/2}}{2}\left(w-\frac{1}{w}\right), \tag{5.6}$$

and so noting the useful relations

$$v^2 = \frac{a}{4w^2}(w^2-1)^2\,, \qquad v^2+a = \frac{a}{4w^2}(w^2+1)^2\,, \tag{5.7}$$

our metric is:

$$ds^2 = \frac{\mu_3 k^2}{2g_s}\frac{(v^2+a)^{1/2}v}{w^2}(dw^2+w^2d\phi^2) = \frac{\mu_3 k^2}{2g_s}\frac{a}{4}\left(1-\frac{1}{w^4}\right)dzd\bar{z}\,, \tag{5.8}$$

where $z = we^{-i\phi}$.

Again, we must match this to the form given in equation 4.13. This time, we can read off the dilaton from the supergravity solution as:

$$e^{-\Phi} = \frac{H_2}{g_s|H_2\cos\phi + i\sin\phi|^2}\,. \tag{5.9}$$

Our crucial equation is now:

$$\left|\frac{\partial Y}{\partial z}\right|^2 = \frac{ak^2}{4}\left((w^2-1)^2\cos^2\phi + (w^2+1)^2\sin^2\phi\right) = \frac{ak^2}{4}\left|1-\frac{1}{z^2}\right|^2, \tag{5.10}$$

where we have used repeatedly that $z = we^{-i\phi}$. It is pleasing that there is a such a simple result, and interesting that although the details seem very different from the example in the previous section, it gives precisely the same form for the change of variables:

$$Y = \frac{a^{1/2}k}{2}\left(z+\frac{1}{z}\right). \tag{5.11}$$

Since H_2 can be written simply as $H_2 = (w^2-1)/(w^2+1)$, once again the algebra simplifies marvellously, and we get the result that $B = z^{-2}$; and therefore the result for the gauge coupling in the natural $\mathcal{N}$=2 adapted complex Y plane is:

$$\begin{aligned}\tau &= \left(\frac{\tau_0 - \bar{\tau}_0 z^2}{1-z^2}\right)\\ &= \frac{i}{g_s}\left(\frac{Y^2}{Y^2 \mp ak^2}\right)^{1/2} + \frac{\theta_s}{2\pi}\,.\end{aligned} \tag{5.12}$$

We have recovered the possibility of the other sign for a. We see that it naturally connects onto the plus sign case as follows: There is again a square root branch cut of width set now by $a^{1/2}k$, and it lies on the real axis for the plus sign choice (corresponding to the minus in the expression immediately above). As a goes to zero, c becomes constant everywhere, and we return to the boring $\mathcal{N}$=4 result. This can be seen by examining the solution for c given in expression 5.1 (with, recall, $\mu = 1$ and $u = 0$ to be on the moduli space, or alternatively $b = 0$ to be asymptotically constant). On the Y-plane this corresponds to the cut shrinking to zero size and disappearing, with τ becoming a constant, τ_0. As a continues to the other sign however, the cut simply reappears, but aligned along the imaginary axis.

6. Discussion

It is intriguing that the results of the previous section for the new solution also give such a clean outcome in the natural $\mathcal{N}$=2 coordinates. There is the same form as in section 4 for the singular locus where the gauge coupling τ diverges. This is a special piece of the moduli space of the full gauge theory. Generically, it is to be expected that there are of order N singular points on the Coulomb branch, and there ought to be nothing special about their arrangement for any N. Gravity and string duals of the gauge theory at large-N suggest the existence of very special large-N limits where these singular points coalesce into a one-dimensional locus. This locus has a stringy and supergravity understanding as the "enhançon" locus of ref. [10]. It is associated with the place where D3-branes (in this example), which would be pointlike in the relevant transverse space, become tensionless and smear out transversely to fill out a one-dimensional (in this example) submanifold densely.

In gauge theory terms, the large-N theory is a special limit in which, away from the singular points, the generic instanton corrections to the form of the coupling are suppressed (due to N being large); but then they switch on strongly at short separation, spreading the points into the singular locus [10, 8].

As we have seen from the probe results, once we get to the right $\mathcal{N}$=2 adapted coordinates the loci seem to be controlled by a very simple structure. Although we have only seen two classes of exact solution for which this can be demonstrated explicitly, they seem so different in the details but yet yield so similar a final form that it is natural to conjecture that this simple structure will persist: The (inverse) square root branch cut form will perhaps always control the location of the locus in this large class of examples given by the supergravity solution in section 2.

We expect that what will distinguish the details on the Coulomb branch is probably only two features: *(1)* The number of distinct such cuts or segments that can appear in the plane — we can imagine multi-cut situations — and *(2)* the detailed distribution of smeared D3-branes within each cut. This will be determined by functions $\rho_i(Y)$ that will give the D3-brane density in the ith segment on the complex Y-plane. These details are implicit in the precise functional dependence of $c(u, v)$, which translates into a specific relation between the mass and the choice of the precise pattern of vacuum expectation values breaking SU(N) to SU($N-1$) $\times$ U(1) (corresponding in the dual theory to pulling off a single D3-brane from the group of N). Perhaps a study along the lines of ref. [8]

could be carried out for new examples to determine the density distributions. Notice that, in the example reviewed in section 4, the density function extracted in ref. [8] was $\rho(Y) \sim \sqrt{m^2 - Y^2}$, which is in fact a semi-circle.

There are a number of suggestive features here, which support a conjecture made some time ago [13] about the presumed underlying simplicity of the physics in question. The conjecture is that there is a reduced model appearing at large N that controls the physics: There may be a matrix model (or closely related integrable model) of some variety responsible for these broad features. Such models at large-N are known to have exactly the attributes required: *(1)* They have simple distributions of eigenvalues or charges, given by density functions $\rho_i(Y)$ (often of Wigner's semi-circle type), sometimes with support on a number of disconnected segments (labelled i here) described in terms of square root branch cuts in the plane. *(2)* Many details of any additional potentials these models might have are irrelevant at strictly large-N, and so they often fall into simple universality classes. The same universality may be at work here. *(3)* Such models are often associated with non-linear differential equations. Perhaps the equation 2.8, seeding the entire class of supergravity solutions, may have its origins in the context of these simpler models.

The matrix model is expected to be closely related to the Calogero-Moser system. The reasons for this expectation are circumstantial, but worth mentioning. The point is that, at any N, the Calogero-Moser model has been shown to share the same formal data as the Seiberg-Witten solution for the associated SU(N) gauge theory [14]. In particular, the Seiberg-Witten curve is essentially the spectral curve of the integrable system defined by the Calogero-Moser model. The Seiberg-Witten curve of course encodes the physics of the Coulomb branch of the gauge theory and, in particular, its points of degeneration give the places where the gauge coupling diverges. The idea [13] would be that at large N this relation becomes more than formal: the dynamics may have their *best* description in terms of the variables of the integrable model. It is also of note that the Calogero-Moser model (in some limits) can be derived from a simple matrix model [15], and that at large-N the distribution of the interacting charges (the eigenvalues) in the model is again Wigner's semi-circle distribution on the line [16]. The description of the $1/N$ corrections may then have their home in familiar matrix model and integrable system technology (see also ref. [17] for related work which may support some of these ideas).

This is very natural also from the point of view of branes, of course. Branes have come to be recognised as having more than a passing resemblance to eigenvalues in some type of matrix model, and this context would be one way to make that precise. This was part of the motivation for the conjecture in ref. [13]; the idea was to find effective models of the enhançon locus as a dynamical object in its own right. The constituent branes have become tensionless and, furthermore, N of them have coalesced into a single unit and so their description in the usual terms is difficult. The expectation was that the sought-after matrix model would be a new collective description of the enhançon. This hope remains. Although the idea seemed somewhat far-fetched at the time, the results presented here together with the new excitement about matrix models' relevance to four dimensional gauge theory [18] suggest that there may be hope to find such a model.

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